Journal of Engineering Mathematics **45:** 39–53, 2003.

Interaction of free-surface waves with a floating dock

This article is dedicated to my very good friend Hendrik Hoogstraten. I enjoyed our first steps in engineering mathematics under the supervision of Rein Timman as much as all the other interests we have in common.

A.J. HERMANS
TU Delft, Faculty of Information Technology and Systems, Department of Applied Mathematical Analysis, Mekelweg 4, 2628 CD Delft, The Netherlands, e-mail: a.j.hermans@its.tudelft.nl

Received 14 August 2002; accepted in revised form 19 September 2002

Abstract. A new method to describe the interaction of waves with a rigid or flexible dock, with zero draft, is derived. By means of Green's theorem an integral equation along the platform for either the velocity potential or the deflection is obtained. In the two-dimensional case this equation is solved by means of a superposition of exponential functions. With a specific choice of the Green function the integration with respect to the space coordinate can be carried out analytically. The integration left is the integration in the k-plane that occurs in the chosen Green function. Subsequently the contour of this integral is modified in the complex plane. This results at first in a dispersion relation for the phase functions in the expansion. Then the set of algebraic equations for the amplitude coefficients follows from the same singularity analysis in the complex plane. These equations are very simple and easy to solve. In contrast to the classical approach of eigen-mode expansions, there is no need to split the problem in a symmetric and antisymmetric one. An other advantage is that the transmission and reflection coefficients are determined seperately by means of Green's theorem, applied at the free surface in the far field. The method is first explained for the semi-infinite rigid dock, followed by the rigid strip, the moving strip and the flexible moving platform. In the appendix it is explained how to derive a set of algebraic equations in the case when the incident wave is not perpendicular to the strip.

Key words: boundary-integral equations, free-surface flows, hydorelasticity, rigid platform, wave diffraction.

1. Introduction

We consider the two-dimensional interaction of an incident wave with a floating dock with small draft. The water depth is finite. This is a classical problem. For instance, Mei and Black [1] have solved a problem by means of a variational approach. They considered a fixed bottom and fixed free-surface obstacle, so they also covered the case of small draft. After the problem has been split in a symmetric and an antisymmetric one, the method consists of matching of eigenfunction expansions of the velocity potential and its normal derivative at the boundaries of two regions. In principle, their method can be extended to the flexible platform case. Because we consider objects with a small draft only, a simpler method can be derived for both the moving rigid and the flexible dock. It is explained by Hermans [2] that in the flexible case the results may serve as the solution of a *canonical* problem for the application of the *ray* method for some three-dimensional problems with a inhomogeneous distribution of the rigidity coefficient. Recently Linton [3] also considered the problem of a fixed two-dimensional small draft dock. He sets up his equations in the same way as done by Mei and Black [1], but his analysis of the equations is different. He uses the modified residue calculus technique to solve the algebraic set of equations. However, as stated in his conclusions, he finally has to solve a set of algebraic equations numerically. The difference with our approach is that we derive an equation at the dock surface directly, while the required continuity of the potential function and velocities is fulfilled automatically.

The rigid-dock problem is less complicated than the flexible-platform problem, so it serves as an interesting problem to explain some details of the method to generate solutions with simple means. We start with the derivation of an integral equation and show that its solution can be written as a superposition of exponential functions with unknown phase and amplitude. If one chooses the Green's function in a proper way, the dispersion relation for the phase functions and a simple set of algebraic equations for the amplitudes are obtained by analytic manipulations in the complex plane. It is also shown that, if one allows the dock to heave and pitch freely, the amplitudes of these motions are easily obtained as well. In this case the method is essentially different from the usually used approach, where one treats excitation and reaction potentials seperately. In our formulation we consider the total potential. The added mass and wave damping are incorporated automatically. In the last section we repeat the same steps to obtain a formulation for the deviation of a flexible platform and also present some results for this case.

2. Mathematical formulation

In this section we derive the general three-dimensional formulation for the diffraction of waves by a thin free-surface obstacle of general geometric form.

The fluid is incompressible, so we introduce the velocity potential $\Phi(x, t)$, such that $\boldsymbol{V}(\boldsymbol{x}, t) = \nabla\Phi(\boldsymbol{x}, t)$, where $\boldsymbol{V}(\boldsymbol{x}, t)$ is the fluid-velocity vector. We assume waves in water without current. Hence, $\Phi(\boldsymbol{x}, t)$ is a solution of the Laplace equation

$$\Delta\Phi = 0 \qquad \text{in the fluid,} \tag{1}$$

together with the linearised kinematic condition, $\Phi_z = \tilde{\zeta}_t$, and dynamic condition, $P/\rho = -\Phi_t - g\tilde{\zeta}$, at the linearized free water surface $z = 0$, where $\tilde{\zeta}(x, y, t)$ denotes the free-surface elevation, and ρ is the density of the water. The linearised free-surface condition outside the obstacle becomes:

$$\frac{\partial^2\Phi}{\partial t^2} + g\frac{\partial\Phi}{\partial z} = 0 \tag{2}$$

at $z = 0$ and $(x, y) \in \mathcal{F}$. The water depth h is assumed to be constant and finite; hence we have $\Phi_z = 0$ at $z = -h$. At first the rigid platform is assumed to be at a fixed position. Later the method is extended for a two-dimensional platform that is free to move in heave and pitch. In the first case the vertical velocity Φ_z at the platform $\mathcal{P}$ becomes zero, while, if the platform is free to move, the vertical velocity is expressed in the unknown heave and pitch motions. The definition of the geometry is presented in Figure 1. Harmonic waves can be written as $\Phi(\boldsymbol{x}, t) = \phi(\boldsymbol{x})\mathrm{e}^{-i\omega t}$ and the undisturbed incident wave equals:

$$\phi^{\text{inc}}(\boldsymbol{x}) = \frac{g\zeta^\infty}{\mathrm{i}\omega}\frac{\cosh(k_0(z+h))}{\cosh(k_0 h)}\exp\{\mathrm{i}k_0(x\ \cos\beta + y\sin\beta)\}, \tag{3}$$

where ζ^∞ is the wave amplitude, ω the frequency, h the water depth, β the angle of incidence with respect to the x-axis, while the real wave number k_0 obeys the dispersion relation,

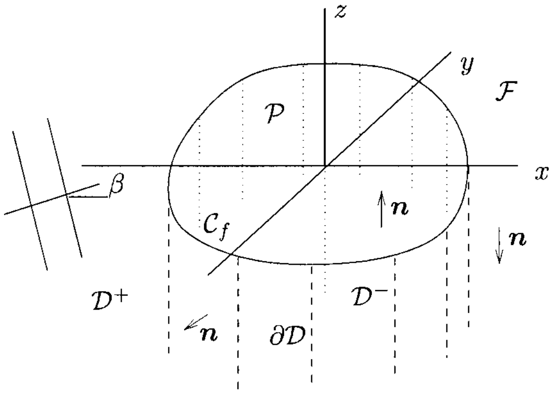


Figure 1. Definition of the geometry.

$$k_0 \tanh(k_0 h) = K = \frac{\omega^2}{g}, \tag{4}$$

for finite water depth.

Next, we derive an integral equation for $\phi(x)$. To do so, the fluid domain will be split up in two regions. We define the region underneath the platform as $\mathcal{D}^-$ and the region towards infinity as $\mathcal{D}^+$, while the interface is denoted by $\partial\mathcal{D}$. The potential function in $\mathcal{D}^+$ is written as a superposition of the incident wave potential and a diffracted wave potential, as follows

$$\phi(\boldsymbol{x}) = \phi^{\text{inc}}(\boldsymbol{x}) + \phi^+(\boldsymbol{x}), \tag{5}$$

while the total potential in $\mathcal{D}^-$ is denoted by $\phi^-(\boldsymbol{x})$. It will be shown that this choice leads to an interesting way to derive an integral equation that can be solved numericaly. At the dividing surface $\partial\mathcal{D}$ we require continuity of the total potential and its normal derivative.

We introduce the Green's function $\mathcal{G}(\boldsymbol{x}, \boldsymbol{\xi})$ that fulfils $\Delta\mathcal{G} = 4\pi\delta(\boldsymbol{x}-\boldsymbol{\xi})$, the boundary conditions at the free-surface and the bottom and also the radiation condition. Green's functions for several free-surface problems can be found in Wehausen and Laitone [4].

We apply Green's theorem first to the potential $\phi^+(\boldsymbol{x})$ and $\mathcal{G}(\boldsymbol{x};\boldsymbol{\xi})$ and then to $\phi^-(\boldsymbol{x})$ and $\mathcal{G}(\boldsymbol{x};\boldsymbol{\xi})$. This leads to the following approach. For $\boldsymbol{x} \in \mathcal{D}^+$ we have:

$$\begin{aligned} 4\pi\phi^+(\boldsymbol{x}) &= -\int_{\partial\mathcal{D}\cup\mathcal{F}} \left(\phi^+(\boldsymbol{\xi})\frac{\partial\mathcal{G}(\boldsymbol{x};\boldsymbol{\xi})}{\partial n} - \mathcal{G}(\boldsymbol{x};\boldsymbol{\xi})\frac{\partial\phi^+(\boldsymbol{\xi})}{\partial n}\right) \mathrm{d}S\,, \\ 0 &= \int_{\partial\mathcal{D}\cup\mathcal{P}} \left(\phi^-(\boldsymbol{\xi})\frac{\partial\mathcal{G}(\boldsymbol{x};\boldsymbol{\xi})}{\partial n} - \mathcal{G}(\boldsymbol{x};\boldsymbol{\xi})\frac{\partial\phi^-(\boldsymbol{\xi})}{\partial n}\right) \mathrm{d}S \end{aligned} \tag{6}$$

and in the region $\boldsymbol{x} \in \mathcal{D}^-$ we have:

$$\begin{aligned} 0 &= -\int_{\partial\mathcal{D}\cup\mathcal{F}} \left(\phi^+(\boldsymbol{\xi})\frac{\partial\mathcal{G}(\boldsymbol{x};\boldsymbol{\xi})}{\partial n} - \mathcal{G}(\boldsymbol{x};\boldsymbol{\xi})\frac{\partial\phi^+(\boldsymbol{\xi})}{\partial n}\right) \mathrm{d}S\,, \\ 4\pi\phi^-(\boldsymbol{x}) &= \int_{\partial\mathcal{D}\cup\mathcal{P}} \left(\phi^-(\boldsymbol{\xi})\frac{\partial\mathcal{G}(\boldsymbol{x};\boldsymbol{\xi})}{\partial n} - \mathcal{G}(\boldsymbol{x};\boldsymbol{\xi})\frac{\partial\phi^-(\boldsymbol{\xi})}{\partial n}\right) \mathrm{d}S\,. \end{aligned} \tag{7}$$

The integrals over $\mathcal{F}$ become zero, due to the zero-current free-surface condition for $\mathcal{G}$ and ϕ^+. We add up the two expressions in (7), for $\boldsymbol{x} \in \mathcal{D}^-$, and use the free-surface condition for the Green's function and the potential ϕ^+. This leads to

$$4\pi\phi^-(\boldsymbol{x}) = \int_{\partial\mathcal{D}} \left([\phi](\boldsymbol{\xi}) \frac{\partial \mathcal{G}(\boldsymbol{x};\boldsymbol{\xi})}{\partial n} - \mathcal{G}(\boldsymbol{x};\boldsymbol{\xi})[\frac{\partial\phi}{\partial n}](\boldsymbol{\xi}) \right) \,\mathrm{d}S + \int_{\mathcal{P}} \left(K\phi^-(\boldsymbol{\xi}) - \frac{\partial\phi^-(\boldsymbol{\xi})}{\partial z} \right) \mathcal{G}(\boldsymbol{x};\boldsymbol{\xi}) \,\mathrm{d}S, \tag{8}$$

where we have used the notation $[\cdots]$ for the jump $\Psi^- - \Psi^+$ of the function Ψ concerned. Furthermore, we use the jump condition between the potentials ϕ^+ and ϕ^- and their derivatives. For the total potential the jumps are zero. The first integral can be further simplified. It is independent of the platform, hence it equals $4\pi\phi^{\text{inc}}$. For $\boldsymbol{x} \in \mathcal{D}^+$ we add up the two expressions in (6) and use expression (5) to arrive at the following expression valid in the whole fluid domain:

$$4\pi\phi(\boldsymbol{x}) = 4\pi\phi^{\text{inc}}(\boldsymbol{x}) + \int_{\mathcal{P}} \left(K\phi(\boldsymbol{\xi}) - \frac{\partial\phi(\boldsymbol{\xi})}{\partial z} \right) \mathcal{G}(\boldsymbol{x};\boldsymbol{\xi}) \,\mathrm{d}S. \tag{9}$$

In the two-dimensional case, that is in the (x, z)-plane, the expression for the total potential becomes:

$$2\pi\phi(x,z) = 2\pi\phi^{\text{inc}}(x,z) + \int_{\mathcal{P}} \left(K\phi(\xi,0) - \frac{\partial\phi(\xi,0)}{\partial z} \right) \mathcal{G}(x,z;\xi,0) \,\mathrm{d}\xi. \tag{10}$$

There are several descriptions for the Green's function available in Wehausen and Laitone [4] for the finite-water-depth case. Most of them are written as a superposition of an infinite fluid source in the fluid region and one mirrored with respect to the bottom. In the deep-water case it is common to combine the source with its mirror with respect to the free surface. In all cases the remainder can be found by means of Fourier transforms in the two horizontal coordinates. For direct numerical computations it is a matter of taste how to split up the Green's function. For our purpose, however, one has to choose to write the Green's function as an infinite fluid source and subtract its mirror with respect the free surface. The remainder follows by Fourier transformations. For the finite-depth case this is a very inconvenient way to compute the perturbed field due to a surface-piercing body. However, in our case we end up with a simple integral because both the source and the collocation points are located at the free surface.

At $z = \zeta = 0$ the two-dimensional Green's function for finite water depth, obeying the radiation condition, has the form:

$$\mathcal{G}(x,0;\xi,0) = -\int_{\mathcal{L}'} \frac{\cosh kh}{k \sinh kh - K \cosh kh} \mathrm{e}^{\mathrm{i}k(x-\xi)} \,\mathrm{d}k \tag{11}$$

and the three-dimensional version has the form:

$$\mathcal{G}(x,y,0;\xi,\eta,0) = -2\int_0^\infty \frac{k \cosh kh}{k \sinh kh - K \cosh kh} J_0(kR) \,\mathrm{d}k. \tag{12}$$

The contour of integration $\mathcal{L}'$ in (11) is given in Figure 2, where k_0 is defined in (4). The contour of integration in (12) is the right-hand part of $\mathcal{L}'$. It is chosen such that the radiation condition is fulfilled. R is the horizontal distance, so $R^2 = (x-\xi)^2 + (y-\eta)^2$.

3. Fixed rigid platform

Here we consider the two-dimensional case and an angle of incidence $\beta = 0$, while in the appendix the extension for $\beta \neq 0$ is explained. In this section the motion of the platform is zero; hence $\phi_z = 0$ at $\mathcal{P}$. If we take $z = 0$ in (10), we arrive at an integral equation:

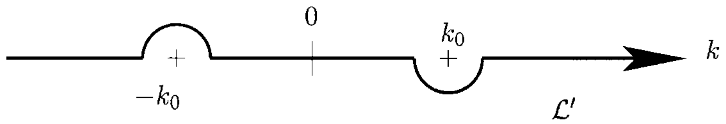


Figure 2. Contour of integration.

$$2\pi\phi(x,0) = 2\pi\phi^{\text{inc}}(x,0) + K\int_{\mathcal{P}} \phi(\xi,0)\mathcal{G}(x,0;\xi,0)\,\mathrm{d}S \quad \text{for } x \in \mathcal{P}. \tag{13}$$

One must realize that the kernel of the integral equation has a weak singularity, so the factor in the left-hand side remains unaltered. In the two-dimensional case this integral equation, with a convenient choice of the Green's function, can be simplified analytically into a matrix equation.

We first consider the half-plane problem, so the platform is present for positive values of x, while the free surface is defined for negative values of x. We seek a solution as a superposition of exponential functions of the form:

$$\phi(x,0) = \frac{g\zeta^{\infty}}{\mathrm{i}\omega}\sum_{n=1}^{\infty} a_n \mathrm{e}^{\mathrm{i}\kappa_n x}. \tag{14}$$

The constant values of the **'amplitudes'** a_n and **'wave numbers'** κ_n will be determined by solving the integral equation. We truncate the series at $n = N$. The values of κ_n will follow from a pole analysis. Obviously, they will be the same as one may obtain by an eigenfunction expansion of the function $\phi(x,z)$. Due to the fact that (14) is an expansion of $\phi(x,0)$, we do not permit ourselves to make use of this *a priori* knowledge. The only requirement, on κ_n, we have at this moment is that, if it is a complex number, the imaginary part must be positive and, for real values, if any, it must be positive due to the radiation condition.

We now insert (14) and (12) into the integral equation (9). We carry out the integration with respect to ξ. The contribution of $x = \infty$ has to be zero; this can be achieved by introducing a small damping in the formulation. For the strip problem this mathematical inconvenience does not occur. We obtain:

$$\sum_{n=1}^{N} a_n \mathrm{e}^{\mathrm{i}\kappa_n x} = \mathrm{e}^{\mathrm{i}k_0 x} + \mathrm{i}\sum_{n=1}^{N} a_n \frac{K}{2\pi}\int_{\mathcal{L}'} \frac{\cosh kh\,\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}k}{(k\sinh kh - K\cosh kh)(k-\kappa_n)}. \tag{15}$$

The poles of the integrand are $k = \kappa_n$, their values still being unknown at this stage, and the zeros of the dispersion relation for the water surface is given as:

$$k\tanh kh = K. \tag{16}$$

This equation has two real solutions, $k = \pm k_0$, and infinitely many along the imaginary axis $k = \pm k_n = \pm \mathrm{i}k_n^{(i)}$. These poles lead to the relations for the determination of the amplitudes a_n.

The values of x are positive, so we may close the contour in the upper half-plane. We assume that the κ_n's are on the real axis or in the upper complex plane. If there are real κ_n's, as will be the case in the elastic case, the radiation condition tells us that, if they are on the positive axis, we have to deform the contour underneath and for the negative values above. Application of the residue lemma then leads to the 'dispersion' relation for κ_n:

$$\sinh\kappa_n h = 0 \tag{17}$$

with solution $\kappa_n h = n\pi\mathrm{i}$ for $n = 1, 2, \cdots$. It is easy to show that the solution $\kappa_0 = 0$ does not contribute.

We now consider the zeros of the dispersion relation for the water surface. These poles lead to the relations for the determination of the amplitudes a_n, by comparing the exponential functions in the final expression and taking their coefficients equal zero. One must realize that the inhomogeneous term in Equation (15) consists of one exponential term, so the only inhomogeneous equation is generated by the pole k_0.

We truncate the series in the ray expansion at N terms, which means that we have to take into account N zeros of the water dispersion relation, one on the real and $N - 1$ on the imaginary axis. If one closes the contour in the complex k-plane, the contribution of these poles leads to:

$$k_0 \sum_{n=1}^{N} \frac{a_n}{\kappa_n - k_0} = Kh - 1 - \frac{k_0^2 h}{K} \tag{18}$$

and for $i = 1, \cdots, N - 1$:

$$\sum_{n=1}^{N} \frac{a_n}{\kappa_n - k_i} = 0. \tag{19}$$

This set of equations is solved numerically for the amplitude coefficients a_n. The pressure distribution along the platform is easily calculated. Convergence tests are carried out by varying the value of N. It turns out that the series expansion has the same nice properties, *e.g.*, fast convergence, as were experienced by Linton [3] and that N can be taken small. This is not surprising because the series expansion for $x \in \mathcal{P}$ is the same as the one one gets in the mode-expansion approach; however, the algebraic equations are completely different.

$$P(x,t) = p(x)\mathrm{e}^{-\mathrm{i}\omega t} = g\zeta^{\infty}\rho \sum_{n=1}^{N} a_n \exp\{\mathrm{i}\kappa_n x - i\omega t\}. \tag{20}$$

An extension of this approach to a strip of finite width l, $0 \le x \le l$, can de done straightforwardly.

$$\phi(x,0) = \frac{g\zeta^{\infty}}{\mathrm{i}\omega} \sum_{n=1}^{N} \left(a_n \mathrm{e}^{\mathrm{i}\kappa_n x} + b_n \mathrm{e}^{-\mathrm{i}\kappa_n (x-l)}\right). \tag{21}$$

We insert this expression into the integral equation (13) and obtain the following relation for $x \in \mathcal{P}$:

$$\begin{aligned} \sum_{n=1}^{N} \left(a_n \mathrm{e}^{\mathrm{i}\kappa_n x} + b_n \mathrm{e}^{-\mathrm{i}\kappa_n (x-l)}\right) &= \mathrm{e}^{\mathrm{i}k_0 x} - \frac{\mathrm{i}K}{2\pi} \int_{\mathcal{L}'} \frac{\cosh kh}{k \sinh kh - K \cosh kh} \\ &\times \sum_{n=1}^{N} \left[\frac{a_n}{k - \kappa_n} \left(\mathrm{e}^{-\mathrm{i}l(k-\kappa_n)} - 1\right) + \frac{b_n}{k + \kappa_n} \left(\mathrm{e}^{-\mathrm{i}kl} - \mathrm{e}^{\mathrm{i}\kappa_n l}\right) \right] \mathrm{e}^{\mathrm{i}kx}\, \mathrm{d}k. \end{aligned} \tag{22}$$

We carry out the same analysis in the complex plane as before. We split the integral in two parts and close the contour in the upper half plane if the k-dependent exponential function $\mathrm{e}^{\mathrm{i}kx}$ occurs and in the lower half plane if $\mathrm{e}^{\mathrm{i}k(x-l)}$ occurs. Because κ_n fulfils (17) the poles $k = \pm\kappa_n$ again give rise to an identity. This can be verified by inspection. We finally obtain $2N$ equations for the unknown a_n and b_n:

$$k_0 \sum_{n=1}^{N} \left[\frac{a_n}{\kappa_n - k_0} - \frac{b_n e^{i\kappa_n l}}{\kappa_n + k_0} \right] = Kh - 1 - \frac{k_0^2 h}{K}$$

and

$$\sum_{n=1}^{N} \left[\frac{-a_n e^{i\kappa_n l}}{\kappa_n + k_0} + \frac{b_n}{\kappa_n - k_0} \right] = 0.$$

For $i = 1, \cdots, N-1$ we obtain:

$$\sum_{n=1}^{N} \left[\frac{a_n}{\kappa_n - k_i} - \frac{b_n e^{i\kappa_n l}}{\kappa_n + k_i} \right] = 0$$

and

$$\sum_{n=1}^{N} \left[\frac{-a_n e^{i\kappa_n l}}{\kappa_n + k_i} + \frac{b_n}{\kappa_n - k_i} \right] = 0.$$

These equations can be solved numerically. The reflection and transmission coefficients follow from the expression for the potential function for $x \in \mathcal{F}$ as given in (13):

$$\begin{aligned} \phi(x,0) &= \phi^{\text{inc}}(x,0) - \frac{gK\zeta^\infty}{2\pi\omega} \int_{\mathcal{L}'} \frac{\cosh kh}{k \sinh kh - K \cosh kh} \\ &\times \sum_{n=1}^{N} \left[\frac{a_n}{k - \kappa_n} \left(e^{-il(k-\kappa_n)} - 1 \right) + \frac{b_n}{k + \kappa_n} \left(e^{-ikl} - e^{i\kappa_n l} \right) \right] e^{ikx} \, dk. \end{aligned} \tag{23}$$

The amplitude of the transmitted wave, $x > l$, can be obtained by closing of the path of integration in the upper half-plane, while the amplitude of the reflected wave, $x < 0$, is obtained by closing the path of integration in the lower half-plane. The contributions in the far field (large values of $|x|$) originate from the poles at $k = k_0$ and $k = -k_0$, respectively. The complex wave elevation in the far field for the reflected wave is $\zeta(x) = \frac{i\omega}{g}\phi(x,0) = R\zeta^\infty e^{-ik_0 x}$ and for the transmitted wave it is written as $\zeta(x) = T\zeta^\infty e^{ik_0 x}$. We obtain for the reflection and transmission coefficients:

$$R = \frac{Kk_0}{(K(1-Kh) + k_0^2 h)} \sum_{n=1}^{N} \left[\frac{a_n}{k_0 + \kappa_n} \left(e^{i(k_0+\kappa_n)l} - 1 \right) + \frac{b_n}{k_0 - \kappa_n} \left(e^{ik_0 l} - e^{i\kappa_n l} \right) \right] \tag{24}$$

and

$$T = 1 + \frac{Kk_0}{(K(1-Kh) + k_0^2 h)} \sum_{n=1}^{N} \left[\frac{a_n}{k_0 - \kappa_n} \left(e^{-i(k_0-\kappa_n)l} - 1 \right) + \frac{b_n}{k_0 + \kappa_n} \left(e^{-ik_0 l} - e^{i\kappa_n l} \right) \right]. \tag{25}$$

The poles in the integrand of (23) along the imaginary axis furnish us with the coefficients of the evanescent modes.

4. Moving rigid platform

In this section we consider a two-dimensional platform that is free to move in heave and pitch. The incident waves are perpendicular to the strip. The mass of the platform, of length l, is denoted by M and the moment of inertia around the center of the platform, $x = l/2$, by I. The heave-and-pitch (positive counter clockwise) motions of the platform are denoted by $W(t) = we^{-\mathrm{i}\omega t}$ and $\tilde{\theta}(t) = \theta e^{-\mathrm{i}\omega t}$. These motions are assumed to be small, so that the vertical motion and velocity of the fluid underneath the platform ($0 \leq x \leq l$) can be approximated by

$$\tilde{\zeta}(x,t) = W(t) + (x - l/2)\tilde{\theta}(t), \quad \zeta(x) = w + (x - l/2)\theta,$$
$$\Phi_z(x,0,t) = \frac{\partial\tilde{\zeta}(x,t)}{\partial t}, \quad \phi_z(x,0) = -\mathrm{i}\omega\zeta(x). \tag{26}$$

The pressure underneath the platform can be approximated by:

$$\frac{p(x)}{\rho} = \mathrm{i}\omega\phi(x,0) - g\,(w + (x - l/2)\theta)\,. \tag{27}$$

In the frequency domain the equations of motion of the platform become:

$$\omega^2 M w = -\int_0^l p(x)\,\mathrm{d}x, \quad \omega^2 I\theta = -\int_0^l (x - l/2)\;p(x)\,\mathrm{d}x. \tag{28}$$

If we insert relation (21) into the expression for the pressure, these equations of motion result in *two* relations between the unknowns $a_n,\ b_n,\ w$ and θ. The two equations of motion lead to the following relations:

$$\sum_{n=1}^{N}\left[\frac{a_n + b_n}{\mathrm{i}\kappa_n}\left(\mathrm{e}^{\mathrm{i}\kappa_n l} - 1\right)\right] + \left[\frac{KM}{\rho} - l\right]\frac{w}{\zeta^\infty} = 0$$

and

$$\sum_{n=1}^{N}(a_n - b_n)\left[\frac{1}{\kappa_n^2}\left(\mathrm{e}^{\mathrm{i}\kappa_n l} - 1\right) + \frac{1}{\mathrm{i}\kappa_n}\left(\mathrm{e}^{\mathrm{i}\kappa_n l} + 1\right)\right] + \left[\frac{KI}{\rho} - \frac{l^3}{12}\right]\frac{\theta}{\zeta^\infty} = 0.$$

The integral equation for the potential function ϕ now becomes:

$$2\pi\phi(x,0) = 2\pi\phi^{\mathrm{inc}}(x,0) + \int_{\mathcal{P}}\left\{K\phi(\xi,0) + \mathrm{i}\omega\,(w + (\xi - l/2)\theta)\right\}\mathcal{G}(x,0;\xi,0)\,\mathrm{d}S \text{ for } \boldsymbol{x} \in \mathcal{P}. \tag{29}$$

We now repeat the same analysis as before and obtain for the $2N$ equations for the unknown $2N + 2$ unknowns $a_n,\ b_n$, w and θ:

$$k_0\sum_{n=1}^{N}\left[\frac{a_n}{\kappa_n - k_0} - \frac{b_n\mathrm{e}^{\mathrm{i}\kappa_n l}}{\kappa_n + k_0}\right] + \frac{1}{\zeta^\infty}\left[w + \theta\left(\frac{1}{\mathrm{i}k_0} - \frac{l}{2}\right)\right] = Kh - 1 - \frac{k_0^2 h}{K}$$

and

$$k_0\sum_{n=1}^{N}\left[\frac{-a_n\mathrm{e}^{\mathrm{i}\kappa_n l}}{\kappa_n + k_0} + \frac{b_n}{\kappa_n - k_0}\right] + \frac{1}{\zeta^\infty}\left[w - \theta\left(\frac{1}{\mathrm{i}k_0} - \frac{l}{2}\right)\right]\mathrm{e}^{\mathrm{i}k_0 l} = 0.$$

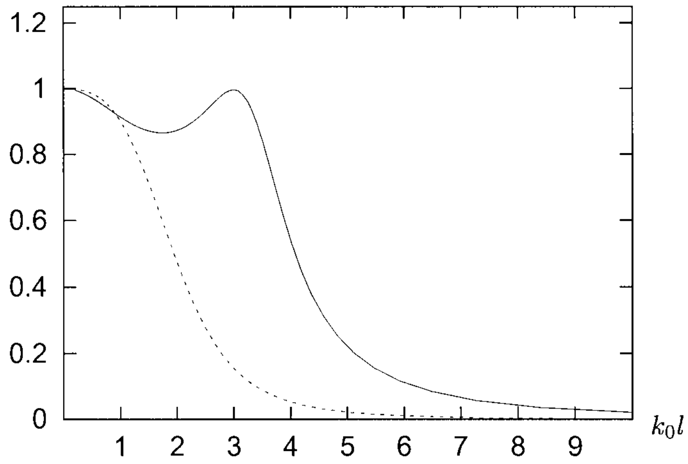


Figure 3. Amplitude of the heave, ____ w/ζ^{∞} and pitch motion, ... $\theta/(k_0\zeta^{\infty})$.

For $i = 1, \cdots, N-1$ we obtain:

$$k_i \sum_{n=1}^{N} \left[\frac{a_n}{\kappa_n - k_i} - \frac{b_n e^{i\kappa_n l}}{\kappa_n + k_i} \right] + \frac{1}{\zeta^{\infty}} \left[w + \theta \left(\frac{1}{ik_i} - \frac{l}{2} \right) \right] = 0$$

and

$$k_i \sum_{n=1}^{N} \left[\frac{-a_n e^{i\kappa_n l}}{\kappa_n + k_i} + \frac{b_n}{\kappa_n - k_i} \right] + \frac{1}{\zeta^{\infty}} \left[w - \theta \left(\frac{1}{ik_i} - \frac{l}{2} \right) \right] e^{ik_i l} = 0.$$

The solution of this set of equations lead to the heave-and-pitch motions directly. In Figure 3 the amplitudes of the heaveand-pitch motion are shown for a platform with $l = 300$ m, $M/\rho = 3 \times 10^5$ m^2 and $I/\rho = 18 \times 10^9$ m^4, while the water depth is $h = 10$ m. We have taken $N = 10$ in these computations. Numerical tests show that the series converge very fast; this is what one should expect because, away from the endpoints of the dock, we have an expansion in exponentially decaying functions. It turns out that at the end points the convergence is also fast. For the chosen water depth $N = 5$ also gives accurate results. For $N = 1$ the values of heave and pitch are within 1 percent of the converged results. One also notices that the low-frequency limit of the heave-and-pith amplidude approaches the correct value. The quantitities given in Figure 3 must approach the value one at low frequencies.

We can now obtain formulas for the reflection and transmission coefficients by considering the asymptotic wave field at $x = \pm\infty$. The pole analysis leads to the following expressions.

$$R = \frac{K}{(K(1-Kh) + k_0^2 h)} \left\{ k_0 \sum_{n=1}^{N} \left[\frac{a_n}{k_0 + \kappa_n} \left(e^{i(k_0+\kappa_n)l} - 1 \right) + \frac{b_n}{k_0 - \kappa_n} \left(e^{ik_0 l} - e^{i\kappa_n l} \right) \right] + \left[1 - e^{ik_0 l} \right] w - \left[\frac{l}{2} \left(1 + e^{ik_0 l} \right) + \frac{1}{ik_0} \left(1 - e^{ik_0 l} \right) \right] \theta \right\} \quad (30)$$

and

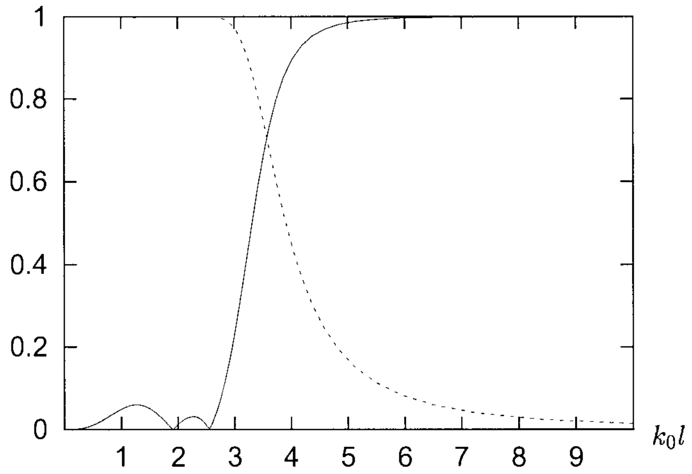


Figure 4. _____ reflection and . . . transmission coefficient.

$$T = \frac{K}{(K(1-Kh)+k_0^2 h)} \left\{ k_0 \sum_{n=1}^{N} \left[\frac{a_n}{k_0-\kappa_n} \left(\mathrm{e}^{-\mathrm{i}(k_0-\kappa_n)l} - 1 \right) + \frac{b_n}{k_0+\kappa_n} \left(\mathrm{e}^{-\mathrm{i}k_0 l} - \mathrm{e}^{\mathrm{i}\kappa_n l} \right) \right] \right.$$
$$\left. + \left[1 - \mathrm{e}^{-\mathrm{i}k_0 l} \right] w - \left[\frac{l}{2} \left(1 + e^{-\mathrm{i}k_0 l} \right) - \frac{1}{\mathrm{i}k_0} \left(1 - \mathrm{e}^{-\mathrm{i}k_0 l} \right) \right] \theta \right\} + 1. \quad (31)$$

An example of the reflection and transmission coefficients is given in Figure 4 for the same parameters as in Figure 3. The energy conservation rule:

$$|R|^2 + |T|^2 = 1$$

is fulfilled in the numerical tests up to seven decimals.

5. Moving flexible platform

In the case of a flexible two-dimensional platform, for $x \in \mathcal{P}$, Green's theorem results in an integral equation for the deflection $W(x,t) = \zeta^\infty w(x) \mathrm{e}^{-\mathrm{i}\omega t}$, for a derivation see Hermans [5],

$$2\pi \left\{ 1 - \mu + \frac{\mathrm{d}^2}{\mathrm{d}x^2} \mathcal{D} \frac{\mathrm{d}^2}{\mathrm{d}x^2} \right\} w(x) + K \int_{\mathcal{P}} \mathcal{G}(x,0;\xi,0) \left\{ \mu - \frac{\mathrm{d}^2}{\mathrm{d}\xi^2} \mathcal{D} \frac{\mathrm{d}^2}{\mathrm{d}\xi^2} \right\} w(\xi)\, \mathrm{d}S = 2\pi \mathrm{e}^{\mathrm{i}k_0 x}, \quad (32)$$

where $\mathcal{D} = \frac{D}{\rho g}$ is the flexural rigidity divided by the density of the water times the acceleration of gravity, $\mu = \frac{mK}{\rho}$ with m the mass of the platform per unit length and width. We follow the same approach as before and introduce:

$$w(x) = \sum_{n=0}^{N+1} \left(a_n \mathrm{e}^{\mathrm{i}\kappa_n x} + b_n \mathrm{e}^{-\mathrm{i}\kappa_n (x-l)} \right). \quad (33)$$

After integration with respect to ξ, the poles in $k = \kappa_n$ lead to the dispersion relation for the elastic plate:

$$(\mathcal{D}\kappa^4 - \mu + 1)\kappa \tanh \kappa h = K. \quad (34)$$

This dispersion relation has solutions in the complex plane, at the real axis $\pm\kappa_0$, at the imaginary axis $\pm\kappa_n,\ n = 3, 4, 5, \cdots$, and four in the complex plane $\pm\kappa_{1,2} = \pm(\kappa_{re} \pm \mathrm{i}\kappa_{im})$. In our expansions only those values that obey the radiation condition play a role; hence the contour of integration passes underneath the poles on the positive real axis and above the one on the negative real axis. One must keep in mind that the position of the poles is similar to that for the dispersion relation for the water surface, except for the two extra complex poles in the upper complex plane.

Again we take into account N solutions for the 'water' dispersion relation (17), while we have $N+2$ zeros of the 'plate' dispersion relation. This leads to $2N$ equations for the $2N+4$ unknowns a_n and b_n.

$$k_0 \sum_{n=0}^{N+1} \left(\mathcal{D}\kappa_n^4 - \mu\right) \left[\frac{a_n}{\kappa_n - k_0} - \frac{b_n \mathrm{e}^{\mathrm{i}\kappa_n l}}{\kappa_n + k_0}\right] = Kh - 1 - \frac{k_0^2 h}{K}$$

and

$$\sum_{n=0}^{N+1} \left(\mathcal{D}\kappa_n^4 - \mu\right) \left[\frac{-a_n \mathrm{e}^{\mathrm{i}\kappa_n l}}{\kappa_n + k_0} + \frac{b_n}{\kappa_n - k_0}\right] = 0.$$

For $i = 1, \cdots, N-1$ we obtain:

$$\sum_{n=0}^{N+1} \left(\mathcal{D}\kappa_n^4 - \mu\right) \left[\frac{a_n}{\kappa_n - k_i} - \frac{b_n \mathrm{e}^{\mathrm{i}\kappa_n l}}{\kappa_n + k_i}\right] = 0$$

and

$$\sum_{n=0}^{N+1} \left(\mathcal{D}\kappa_n^4 - \mu\right) \left[\frac{-a_n \mathrm{e}^{\mathrm{i}\kappa_n l}}{\kappa_n + k_i} + \frac{b_n}{\kappa_n - k_i}\right] = 0.$$

The boundary conditions at the edge of the platform

$$\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} = 0 \ \text{ and } \ \frac{\mathrm{d}^3 w}{\mathrm{d}x^3} = 0 \quad \text{at } \ x = 0, l$$

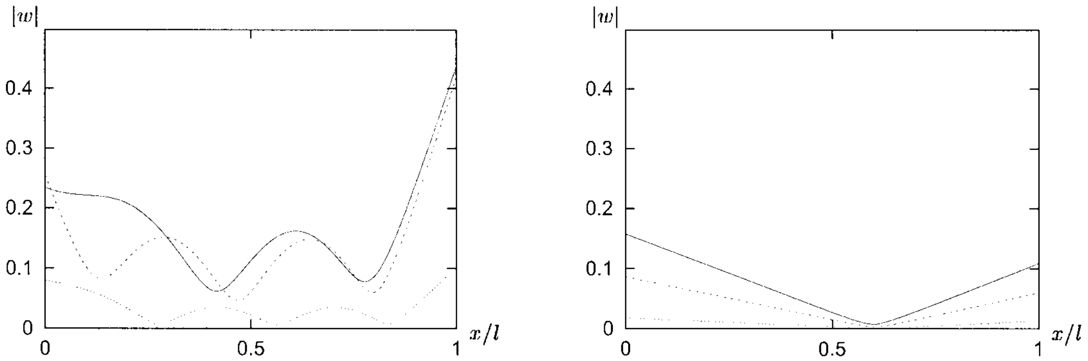
lead to four equations for the unknowns a_n and b_n. The reflection and transmission coefficients now become:

$$R = \frac{Kk_0}{(K(1-Kh) + k_0^2 h)} \sum_{n=0}^{N+1} \left(\mathcal{D}\kappa_n^4 - \mu\right) \left[\frac{a_n}{k_0 + \kappa_n}\left(\mathrm{e}^{\mathrm{i}(k_0+\kappa_n)l} - 1\right) + \frac{b_n}{k_0 - \kappa_n}\left(\mathrm{e}^{\mathrm{i}k_0 l} - \mathrm{e}^{\mathrm{i}\kappa_n l}\right)\right] \tag{35}$$

and

$$T = \frac{Kk_0}{(K(1-Kh) + k_0^2 h)} \sum_{n=0}^{N+1} \left(\mathcal{D}\kappa_n^4 - \mu\right) \left[\frac{a_n}{k_0 - \kappa_n}\left(\mathrm{e}^{-\mathrm{i}(k_0-\kappa_n)l} - 1\right) + \frac{b_n}{k_0 + \kappa_n}\left(\mathrm{e}^{-\mathrm{i}k_0 l} - \mathrm{e}^{\mathrm{i}\kappa_n l}\right)\right] + 1. \tag{36}$$

Numerical results for the deflection are shown in Figures 5(a,b) for $h = 10$ m, $\mathcal{D} = 10^7$ m^4 and 10^{10} m^4 with $\Lambda = 2\pi/K$. The reflection and transmission coefficients for the same waterdepth and $\mathcal{D} = 10^7$ m^4 are shown are Figure 6. In practical cases one is also interested in the second-order mean drft-force in the horizontal direction. According to Maruo [6], see also Hermans [7], the drift-force can be expressed in the ampitude of the reflected wave



a. $\mathcal{D} = 10^7$ m^4 b. $\mathcal{D} = 10^{10}$ m^4

Figure 5. Deflection for $\Lambda/l = 0{\cdot}5$, $0{\cdot}3$, $0{\cdot}1$ (top-down)

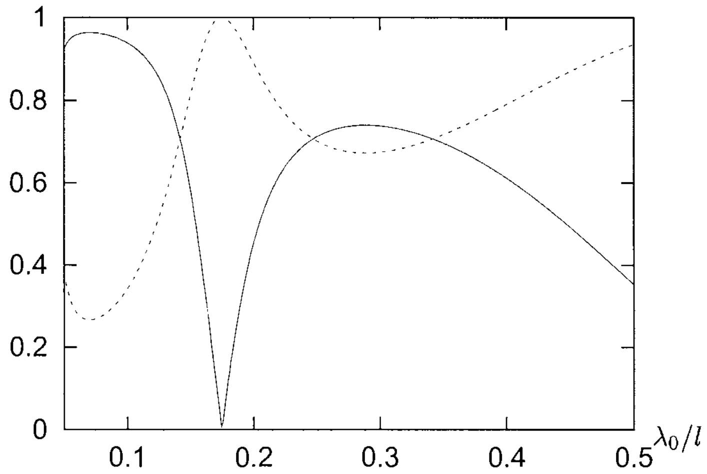


Figure 6. _____ reflection and . . . transmission coefficient.

$$D_x = \frac{1}{2}\rho g|R\zeta^{\infty}|^2.$$

6. Conclusions

It is shown that the two-dimensional shallow-dock problem can be solved by rather simple means. The only approximations made are that we may consider both the free-surface elevation and draft of the dock to be of the same order of magnitude. This results in a boundary-value problem at $z = 0$. With a specific Green's function an integral equation can be formulated with solutions that can be written as a series of exponential functions, similar as eigenfunction expansions. The series may be truncated at a finite number of terms. In the computations shown ten terms are taken into account. Computation with a hundred terms shows that ten terms give a very high degree of accuracy, for a wide range of the water depth h. It is also found that in many cases even one term is sufficient. The well-known dispersion relation for the platform influenced by the water region underneath the platform follows from a pole analysis, while a set of equations for the coefficients of the exponential functions follow from the same singularity analysis. This approach is much more efficient than the Wiener-Hopf formulation suggested by Tkacheva [8] for the flexible platform. The results of both methods are comparable. It also becomes clear that, for the zero-draft dock, there is no reason to split up the problem in a symmetric and an antisymmetric one, as has been done by Linton [3]. The complete problem has been solved at once. The analysis described here can not be carried out direcly in the case of a platform with finite draft.

Hermans [2] has shown that, in the case of short-wave diffraction by a flexible platform with inhomogeneous elastic properties, the problem treated here may serve as a '*canonical*' problem for the application of the '*ray*' method. A direct application of the ray method leads to an incomplete initial-value problem. The *missing* initial conditions are obtained by the method shown here. In the deep-water case, $h = \infty$, it is advised to use the finite-depth formulation with a large value of h compared to the wavelength. The iterative process, for the deep-water case, as described by Hermans [9] is less efficient in comparison with the method described here.

Acknowledgements

The author thanks one of the reviewers for pointing out the existence of the paper of Linton and a missing factor of $\frac{1}{2}$ in expression (A4).

Appendix, Incident waves at obligue angles

Here we give a description for the case of a fixed strip while the angle of incidence $\beta \neq 0$.

$$4\pi\phi(x, y, 0) = 4\pi\phi^{\text{inc}}(x, y, 0) + K\int_{\mathcal{P}} \phi(\xi, \eta, 0)\mathcal{G}(x, y, 0; \xi, \eta, 0)\, \mathrm{d}S \quad \text{for } \boldsymbol{x} \in \mathcal{P}. \tag{A1}$$

We first consider the half-plane problem, so the platform is present for positive values of x, while the free surface is defined for negative values of x. We seek a solution as a superposition of exponential functions of the form:

$$\phi(x, y, 0) = \frac{g\zeta^{\infty}}{\mathrm{i}\omega}\sum_{n=1}^{\infty} a_n \exp\{\mathrm{i}\kappa_n x + ik_0 y\sin\beta\} \quad \text{for } 0 \leq \beta \leq \tfrac{\pi}{2}. \tag{A2}$$

The only requirement, on κ_n, we have at this moment is that, if it is a complex number, the imaginary part must be positive.

We now insert (A2) and (12) in the integral equation (A1). The infinte path of integration with respect to η is changed in the semi-infinite path from 0 to $+\infty$. Then we carry out the integration with respect to ξ and η. To carry out the integration with respect to η we make use of a Sonine-Gegenbauer expressions for Bessel functions:

$$\int_0^{\infty} \cos(bt)J_0\{k\sqrt{a^2 + t^2}\}\, \mathrm{d}t = \begin{cases} 0 & \text{if } k < b \\ \dfrac{\cos(a\sqrt{k^2 - b^2})}{\sqrt{k^2 - b^2}} & \text{if } k > b \end{cases}. \tag{A3}$$

The coefficient b in expression (A3) corresponds to $k_0 \sin\beta$. Finally, the integration with respect to k is written as an integration along $\mathcal{L}'$. The result becomes:

$$\sum_n a_n \mathrm{e}^{\mathrm{i}\kappa_n x} = \mathrm{e}^{\mathrm{i}k_0 x\cos\beta} + \mathrm{i}\sum_n a_n \frac{K}{4\pi}\int_{\mathcal{L}'} \frac{\cosh kh}{k\sinh kh - K\cosh kh}$$
$$\times\left(\frac{\mathrm{e}^{\mathrm{i}x\sqrt{k^2-k_0^2\sin^2\beta}}}{\sqrt{k^2-k_0^2\sin^2\beta}-\kappa_n} - \frac{\mathrm{e}^{-\mathrm{i}x\sqrt{k^2-k_0^2\sin^2\beta}}}{\sqrt{k^2-k_0^2\sin^2\beta}+\kappa_n}\right)\frac{k\,\mathrm{d}k}{\sqrt{k^2-k_0^2\sin^2\beta}} \tag{A4}$$

We now define $\kappa^{(n)}$ as follows:

$$\kappa^{(n)2} = \kappa_n^2 + k_0^2\sin^2\beta.$$

One must take care of the discontinuous behaviour of (A3) properly. This can be done by splitting the integral in two parts:

$$I_1 = \int_{\mathcal{L}'} \frac{\cosh kh\,\mathrm{e}^{\mathrm{i}x\sqrt{k^2-k_0^2\sin^2\beta}}\quad k\,\mathrm{d}k}{(k\sinh kh - K\cosh kh)\left(\sqrt{k^2-k_0^2\sin^2\beta}-\kappa_n\right)\sqrt{k^2-k_0^2\sin^2\beta}} \tag{A5}$$

and

$$I_2 = -\int_{\mathcal{L}'} \frac{\cosh kh\,\mathrm{e}^{-\mathrm{i}x\sqrt{k^2-k_0^2\sin^2\beta}}\quad k\,\mathrm{d}k}{(k\sinh kh - K\cosh kh)\left(\sqrt{k^2-k_0^2\sin^2\beta}+\kappa_n\right)\sqrt{k^2-k_0^2\sin^2\beta}} \tag{A6}$$

The contour $\mathcal{L}'$ passed the branch cut at the upper side in I_1 and at the lower side I_2. This results in a zero contribution of the path along the branch cut in (A4), in accordance with the discontinuous behaviour of the Sonine-Gegenbauer integral. For positive values of x the first part I_1 may be closed in the upper half-plane and the second part I_2 in the lower half-plane.

Until now the values of κ_n are still unknown. We assume that the poles at $\sqrt{k^2-k_0^2\sin^2\beta} = \kappa_n$ are in the upper half-plane. Application of the residue lemma at these points leads to the 'dispersion' relation for $\kappa^{(n)}$:

$$\sinh\kappa^{(n)}h = 0 \tag{A7}$$

with solution $\kappa^{(n)}h = n\pi\mathrm{i}$ for $n = 1, 2, \cdots$. It is easy to show that the solution $\kappa^{(0)} = 0$ does not contribute.

We now consider the zeros of the dispersion relation for the water surface

$$k\tanh kh = K. \tag{A8}$$

This equation has two real solution $k = \pm k_0$ and infinitely many along the imaginary axis $k = \pm k_n = \pm\,\mathrm{i}k_n^{(i)}$. These poles lead to the relations for the determination of the amplitudes a_n.

We truncate the series in the expansion at N terms; this means that we have to take into account N zeros of the water dispersion relation, one on the real axis and $N-1$ imaginary. If one closes the contour in the complex k-plane the contribution of these poles leads to:

$$\frac{Kk_0}{(K(1-Kh)+k_0^2h)\cos\beta}\sum_{n=1}^{N}\frac{a_n}{(\kappa_n - k_0\cos\beta)} + 1 = 0 \tag{A9}$$

and for $i = 1, \cdots, N-1$:

$$\sum_{n=1}^{N} \frac{a_n}{(\kappa_n - \sqrt{k_i^2 - k_0^2 \sin^2 \beta})} = 0. \tag{A10}$$

This set of equations for the amplitude coefficients a_n can be solved among others numerically.

An extension of this approach to a strip of finite width l, $0 \le x \le l$, can de done straightforwardly.

$$\phi(x, y, 0) = \frac{g\zeta^\infty}{\mathrm{i}\omega} \mathrm{e}^{\mathrm{i}k_0 y \sin\beta} \sum_{n=1}^{N} \left(a_n \mathrm{e}^{\mathrm{i}\kappa_n x} + b_n \mathrm{e}^{-\mathrm{i}\kappa_n (x-l)}\right). \tag{A11}$$

If we carry out the same analysis in the complex plane as before, we obtain $2N$ equations for the unknown a_n and b_n:

$$\frac{Kk_0}{(K(1-Kh) + k_0^2 h)\cos\beta} \sum_{n=1}^{N} \left[\frac{a_n}{\kappa_n - k_0\cos\beta} - \frac{b_n \mathrm{e}^{\mathrm{i}\kappa_n l}}{\kappa_n + k_0 \cos\beta}\right] + 1 = 0$$

and

$$\sum_{n=1}^{N} \left[\frac{-a_n \mathrm{e}^{\mathrm{i}\kappa_n l}}{\kappa_n + k_0\cos\beta} + \frac{b_n}{\kappa_n - k_0\cos\beta}\right] = 0.$$

For $i = 1, \cdots, N-1$ we obtain:

$$\sum_{n=1}^{N} \left[\frac{a_n}{\kappa_n - \sqrt{k_i^2 - k_0^2 \sin^2\beta}} - \frac{b_n \mathrm{e}^{\mathrm{i}\kappa_n l}}{\kappa_n + \sqrt{k_i^2 - k_0^2 \sin^2\beta}}\right] = 0$$

and

$$\sum_{n=1}^{N} \left[\frac{-a_n \mathrm{e}^{\mathrm{i}\kappa_n l}}{\kappa_n + \sqrt{k_i^2 - k_0^2 \sin^2\beta}} + \frac{b_n}{\kappa_n - \sqrt{k_i^2 - k_0^2 \sin^2\beta}}\right] = 0.$$

References

1. C.C. Mei and J.L. Black, Scattering of surface waves by rectangular obstacles in waters of finite depth. *J. Fluid Mech.* 38 (1969) 499–511.
2. A.J. Hermans, The ray method for the deflection of a floating flexible platform in short waves. *J. Fluids Struct.* (2003) (accepted for publication)
3. C.M. Linton, The finite dock problem. *Zeitschrift für angewandte Mathematik und Physik (ZAMP)* 52 (2001) 640–656.
4. J.V. Wehausen, and E.V. Laitone, Surface waves. In: S. Flügge (ed.), *Encyclopedia of Physics*, Vol. 9. Berlin: Springer-Verlag (1960) pp. 446–814, also at http://www.coe.berkeley.edu/SurfaceWaves/
5. A.J. Hermans, A boundary element method for the interaction of free-surface waves with a very large floating flexible platform. *J. Fluids Struct.* 14 (2000) 943–956.
6. H. Maruo, (1960). The drift of a body floating in waves. *J. of Ship Research* 4 (1960) 1–10.
7. A.J. Hermans, Low frequency second-order wave-drift forces and damping. *J. Eng. Math.* 35 (1999) 181–198.
8. L.A. Tkacheva, Diffraction of surface waves at floating elastic plate. In: R.C.T. Rainey and S.F. Lee (eds), *Proceedings of the 17th Int. Workshop on Water Waves and Floating Bodies*. Peterhouse, Cambridge, (UK): WSAtkins (2002) 175–179.
9. A.J. Hermans, A geometrical-optics approach for the deflection of a floating flexible platform. *Appl. Ocean Res.* 23 (2002) 269–276.